- $\mathbb{N}$ denotes the set of positive integers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.

Q1. Find the values of $a > 0$ for which the improper integral

$$\int_0^\infty \frac{\sin x}{x^a}\, dx$$

converges.

Q2. Let $f : [0,1] \to \mathbb{R}$ be a real-valued continuous function which is differentiable on $(0,1)$ and satisfies $f(0) = 0$. Suppose there exists a constant $c \in (0,1)$ such that

$$|f'(x)| \leq c|f(x)| \quad \text{for all } x \in (0,1).$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

Q3. Let G be an abelian group of order n.

(a) If $f : G \to \mathbb{C}$ is a function, then prove that for all $h \in G$,

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(hg).$$

(b) Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers and suppose $f : G \to \mathbb{C}^*$ is a homomorphism. Prove that

$$\sum_{g \in G} f(g) = 0 \quad \text{or} \quad \sum_{g \in G} f(g) = n.$$

(c) If $f : G \to \mathbb{C}^*$ is any homomorphism, then prove that

$$\sum_{g \in G} |f(g)| = n.$$

Q4. (a) Is the ideal $I = (X+Y, X-Y)$ in the polynomial ring $\mathbb{C}[X,Y]$ a prime ideal? Justify your answer.

(b) Is the ideal $I = (X+Y, X-Y)$ in the polynomial ring $\mathbb{Z}[X,Y]$ a prime ideal? Justify your answer.

Q5. Let $n \geq 2$ and A be an $n \times n$ matrix with real entries. Let $\operatorname{Adj} A$ denote the adjoint of A, that is, the (i, j)-th entry of $\operatorname{Adj} A$ is the (j, i)-th cofactor of A.

Show that the rank of $\operatorname{Adj} A$ is $0, 1$ or n.

Q6. Suppose an urn contains a red ball and a blue ball. A ball is drawn at random and a ball of the same colour is added to the urn along with the one that was drawn. This process is repeated indefinitely.

Let X denote the random variable that takes the value n if the first $n-1$ draws yield red balls and the n-th draw yields a blue ball.

(a) If $n \geq 1$, find $P(X > n)$.

(b) Show that the probability of a blue ball being chosen eventually is 1.

(c) Find $E[X]$.

Q7. A real number x_0 is said to be a limit point of a set $S \subseteq \mathbb{R}$ if every neighbourhood of x_0 contains a point of S other than x_0. Consider the set

$$S = \{0\} \cup \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\}.$$

(a) Show that S contains infinitely many limit points of S.

(b) Show that S is a compact subset of $\mathbb{R}$.

(c) Find all limit points of S.

Q8. (a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[0, 1]$ such that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $(0, 1]$. Show that $\sum_{n=1}^{\infty} f_n(0)$ converges.

(b) Find the set D of all points $x \in [0, 1]$ such that the series

$$\sum_{n=1}^{\infty} e^{-nx} \cos nx$$

converges. Does this series converge uniformly on D? Justify your answer.